Computer Vision I

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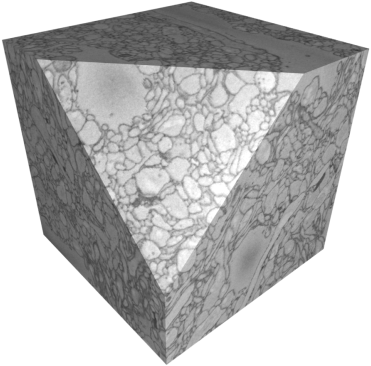
Machine Learning for Computer Vision
TU Dresden

https://mlcv.cs.tu-dresden.de/courses/24-winter/cv1/

Winter Term 2024/2025

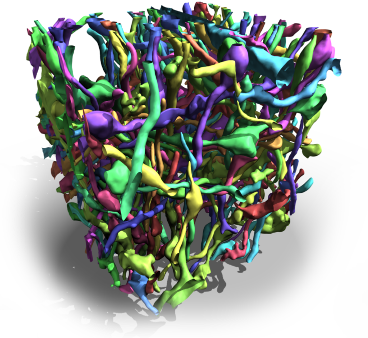
- So far, we have studied **pixel classification**, a problem whose feasible solutions define decisions at the pixels of an image
- Next, we will study **image decomposition**, a problem whose feasible solutions decide whether pairs of pixels are assigned to the same or distinct components of the image
- Image decomposition has applications where components of the image are indistinguishable by appearance (see next slide)

Image decomposition



Volume Image ($\sim$32 nm/voxel)[1]

$\mapsto$



Decomposition[2]

[1] Denk and Horstmann 2004. 10.1371/journal.pbio.0020329

[2] A, Köthe, Kröger, Helmstaedter, Briggman, Denk and Hamprecht 2012. 10.1016/j.media.2011.11.004

Image decomposition

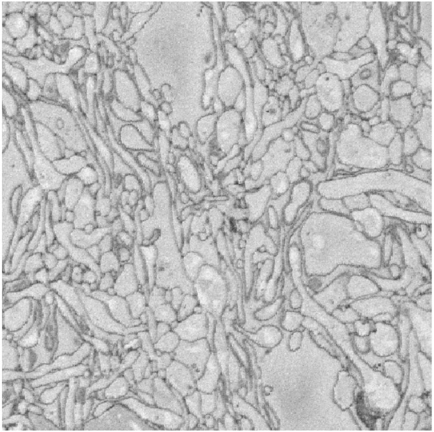
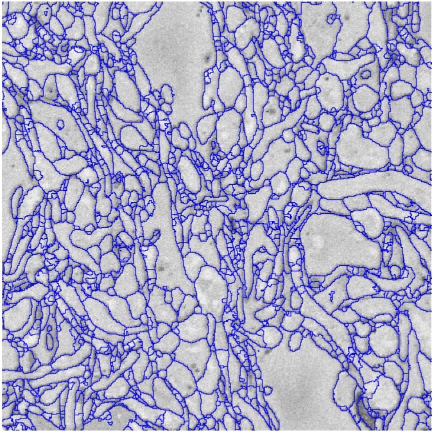


Image decomposition



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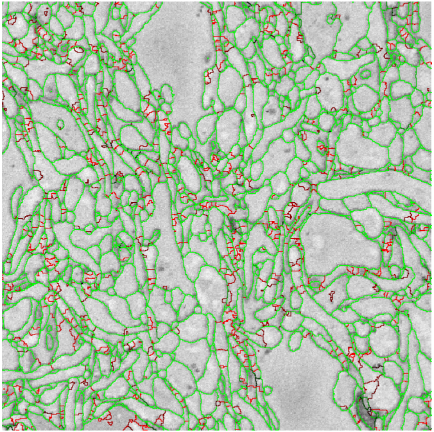

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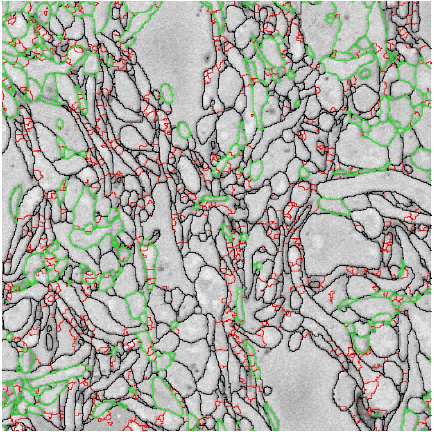
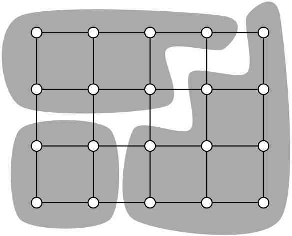
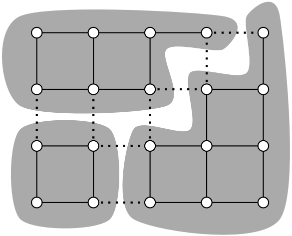




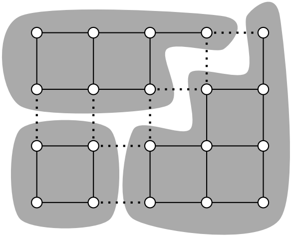
Decomposition of a graph $G = (V, E)$

- A mathematical abstraction of a decomposition of an image is a decomposition of the pixel grid graph.
- A decomposition of a graph is a partition of the node set into connected subsets (one example is depicted above in gray).



Decomposition of a graph $G = (V, E)$

- A decomposition of a graph is characterized by the set of edges that straddle distinct components (depicted above as dotted lines)
- Those subsets of edges are called **multicuts** of the graph



Multicut of a graph $G = (V, E)$

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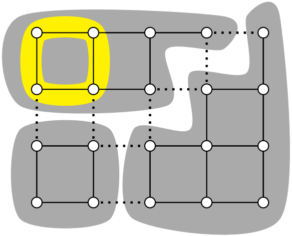
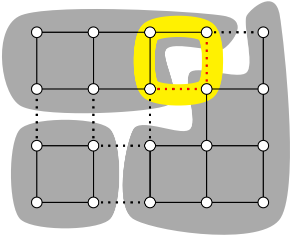


Multicut of a graph $G = (V, E)$

- A subset of edges is a multicuts iff no cycle in the graph intersects with the subset in precisely one edge



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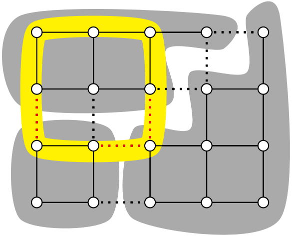

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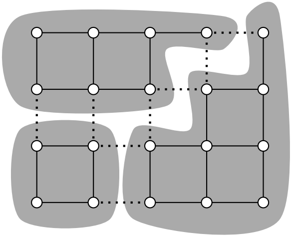
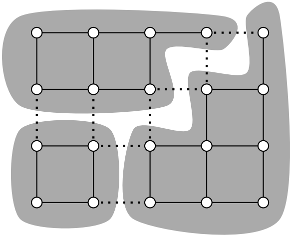


Multicut of a graph $G = (V, E)$

$$\text{multicuts}(G) := \{M \subseteq E \,|\, \forall C \in \text{cycles}(G) :\ |M \cap C| \neq 1\}$$



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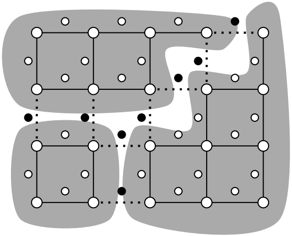
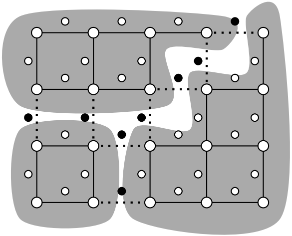


Multicut of a graph $G = (V, E)$

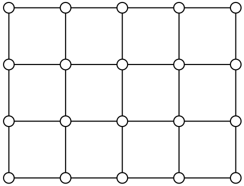
- The characteristic function $y \colon E \to \{0, 1\}$ of a multicut $y^{-1}(1)$ can be used to encode the decomposition induced by the multicut in an $|E|$-dimensional 01-vector
- For any $e \in E$, $y_e = 1$ indicates that an edge is cut, straddling distinct components



Multicut of a graph $G = (V, E)$

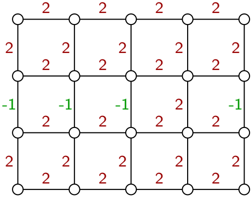
- The set of the characteristic functions of all multicuts of G:

$$Y_G := \left\{ y \in \{0,1\}^E \;\middle|\; \forall (V_C, E_C) \in \mathsf{cycles}(G)\, \forall e \in E_C : \, y_e \leq \sum_{f \in E_C \setminus \{e\}} y_f \right\}$$



Graph $G = (V, E)$

- An instance of the image decomposition problem is given by a graph $G = (V, E)$ and, for every edge $e = \{v, w\} \in E$, a (positive or negative) cost $c_e \in \mathbb{R}$ that is payed iff the incident pixels v and w are put in distinct components
- Such costs can be learned (as described earlier in the course), e.g., $c_e = -f_\theta(x_e)$, or more specifically, $c_e = -\langle \theta, x_e \rangle$.



Graph $G = (V, E)$. Edge costs $c : E \to \mathbb{R}$

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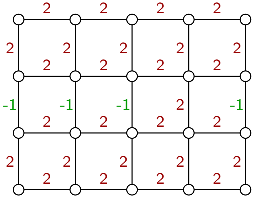


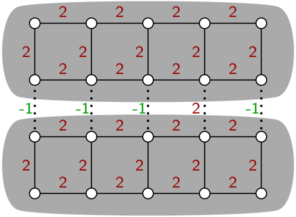
Graph $G = (V, E)$. Edge costs $c : E \to \mathbb{R}$

- Image decomposition problem:

$$\min_{y \in Y_G} \sum_{e \in E} c_e \, y_e$$

- The optimal solution is shown on the next slide

Image decomposition



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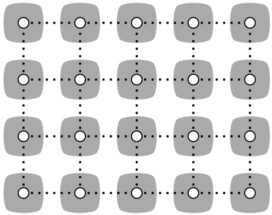

- One technique for finding feasible solutions to an image decomposition problem is **local search**.
- Starting from the finest decomposition into singleton components (depicted above), we greedily join neighboring components as long as this improves the cost (see next slide).

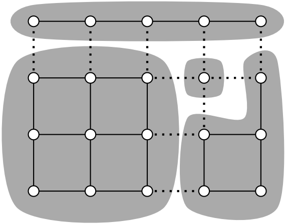

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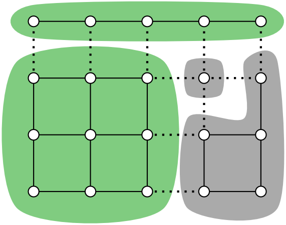
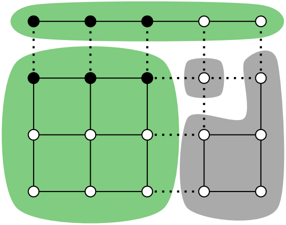
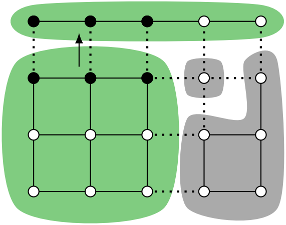
Image decomposition



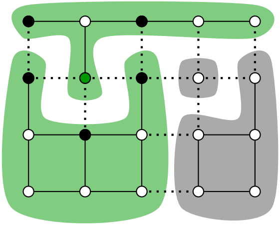
- Once no joining of neighboring components further reduces the cost, we consider all pairs of neighboring components (depicted in green)



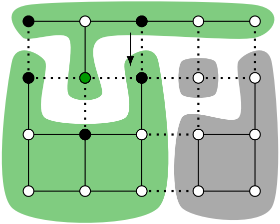
- Once no joining of neighboring components further reduces the cost, we consider all pairs of neighboring components (depicted in green) and all nodes at the shared boundary (depicted in black)



- Once no joining of neighboring components further reduces the cost, we consider all pairs of neighboring components (depicted in green) and all nodes at the shared boundary (depicted in black) and all possibilities of moving nodes from one component to the other.



- Once no joining of neighboring components further reduces the cost, we consider all pairs of neighboring components (depicted in green) and all nodes at the shared boundary (depicted in black) and all possibilities of moving nodes from one component to the other.
- The procedure is iterated until no such transformation further reduces the cost



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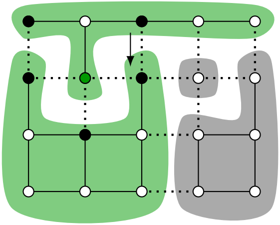

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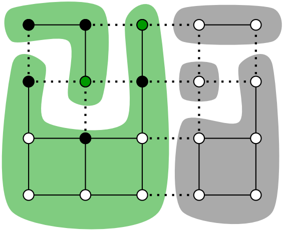

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Definition. Let $G = (V, E)$ be any graph.

- A subgraph $G' = (V', E')$ of G is called a **component (cluster)** of G iff G' is non-empty, node-induced (i.e. $E' = E \cap \binom{V'}{2}$) and connected.
- A partition Π of the node set V is called a **decomposition (clustering)** of G iff, for every $U \in \Pi$, the subgraph $(U, E \cap \binom{U}{2})$ of G induced by U is connected (and thus a component of G).
- Let D_G denote the set of all decompositions of G.

Definition. Let $G = (V, E)$ be any graph. A subset $M \subseteq E$ of edges is called a **multicut** of G iff, for every cycle $C \subseteq E$ of G: $|C \cap M| \neq 1$. Let M_G denote the set of all multicuts of G.

Lemma. For any decomposition of a graph G, the set of those edges that straddle distinct components is a multicut of G. This multicut is said to be **induced** by the decomposition. The map from decompositions to induced multicuts is a **bijection** from D_G to M_G.

Lemma. For any graph $G = (V, E)$ and any $y \in \{0,1\}^E$, the set $y^{-1}(1)$ is a multicut of G iff the following inequalities are satisfied:

$$\forall (V_C, E_C) \in \textsf{cycles}(G)\ \forall e \in E_C\colon \quad y_e \leq \sum_{f \in E_C \setminus \{e\}} y_f \tag{1}$$

Definition. For any graph $G = (V, E)$, any $c \in \mathbb{R}^E$ and the set Y_G of all $y \in \{0,1\}^E$ that satisfy (1), the **minimum cost multicut problem** has the form

$$\min_{y \in \{0,1\}^E} \quad \underbrace{\sum_{e \in E} c_e\, y_e}_{=:\varphi(y)}$$

$$\text{subject to} \quad \forall (V_C, E_C) \in \textsf{cycles}(G)\ \forall e \in E_C\colon \quad y_e \leq \sum_{f \in E_C \setminus \{e\}} y_f$$

Definition. For any graph $G = (V, E)$ and any decomposition Π of G, let y^{Π} the characteristic function of the multicut induced by Π, i.e. $y^{\Pi} \in \{0,1\}^E$ such that

$$\forall e \in E\colon \quad y^{\Pi}_e = 0 \;\Leftrightarrow\; \exists U \in \Pi\colon e \subseteq U \tag{2}$$

Remark: The characteristic function $y \in \{0,1\}^E$ of a multicut $y^{-1}(1)$ makes explicit for every edge $\{u,v\} = e \in E$ whether the incident nodes u and v belong to the same component ($y_e = 0$) or distinct components ($y_e = 1$).

Greedy joining algorithm:

- The greedy joining algorithm is a local search algorithm that starts from any initial decomposition.
- It searches for decompositions with lower cost by joining pairs of **neighboring (!)** components recursively.
- As components can only grow and the number of components decreases by one in every step, one typically starts from the finest decomposition Π_0 into one-elementary components.

Definition. Let $G = (V, E)$ be any graph.

- For any disjoint sets $B, C \subseteq V$, the pair $\{B, C\}$ is called **neighboring** in G iff there exist nodes $b \in B$ and $c \in C$ such that $\{b, c\} \in E$.
- For any decomposition Π of G, we define

$$\mathcal{E}_\Pi = \left\{ \{B, C\} \in \tbinom{\Pi}{2} \,\middle|\, \exists b \in B \, \exists c \in C \colon \{b, c\} \in E \right\} \quad . \tag{3}$$

- For any decomposition Π of G and any $\{B, C\} \in \mathcal{E}_\Pi$, let $\mathsf{join}_{BC}[\Pi]$ be the decomposition of G obtained by joining the sets B and C in Π, i.e.

$$\mathsf{join}_{BC}[\Pi] = (\Pi \setminus \{B, C\}) \cup \{B \cup C\} \quad . \tag{4}$$

$\Pi' =$ greedy-joining(Π)
choose $\{B,C\} \in \underset{\{B',C'\} \in \mathcal{E}_\Pi}{\operatorname{argmin}} \varphi(y^{\mathsf{join}_{B'C'}[\Pi]}) - \varphi(y^\Pi)$
if $\varphi(y^{\mathsf{join}_{BC}[\Pi]}) - \varphi(y^\Pi) < 0$
$\quad \Pi' :=$ greedy-joining(join$_{BC}[\Pi]$)
else
$\quad \Pi' := \Pi$

Greedy moving algorithm:

- The greedy moving algorithm is a local search algorithm that starts from any initial decomposition, e.g., the fixed point of greedy joining.
- It searches for decompositions with lower cost by recursively moving individual nodes from one component to a **neighboring (!)** component, possibly a new one.
- When a **cut node** is moved out of a component or a node is moved to a new component, the number of components increases. When the last node is moved out of a component, the number of components decreases.

Definition. For any graph $G = (V, E)$ and any decomposition Π of G, the decomposition Π is called **coarsest** iff, for every $U \in \Pi$, the component $(U, E \cap \binom{U}{2})$ induced by U is maximal.

Lemma. For any graph G, the coarsest decomposition is unique. We denote it by Π^*_G.

Definition. For any graph $G = (V, E)$, any decomposition Π of G and any $a \in V$, let U_a the unique $U_a \in \Pi$ such that $a \in U_a$, and let

$$\mathcal{N}_a = \{\emptyset\} \cup \{W \in \Pi \,|\, a \notin W \ \wedge \ \exists w \in W \colon \{a, w\} \in E\} \tag{5}$$

$$G_a = \left(U_a \setminus \{a\},\, E \cap \binom{U_a \setminus \{a\}}{2}\right) \tag{6}$$

For any $U \in \mathcal{N}_a$, let $\mathsf{move}_{aU}[\Pi]$ the decomposition of G obtained by moving the node a to the set U, i.e.

$$\mathsf{move}_{aU}[\Pi] = \Pi \setminus \{U_a, U\} \cup \{U \cup \{a\}\} \cup \Pi^*_{G_a} \ . \tag{7}$$

$\Pi' = \text{greedy-moving}(\Pi)$

choose $(a, U) \in \underset{a' \in A,\ U' \in \mathcal{N}_{a'}}{\operatorname{argmin}} \varphi(y^{\mathsf{move}_{a'U'}[\Pi]}) - \varphi(y^{\Pi})$

if $\varphi(y^{\mathsf{move}_{aU}[\Pi]}) - \varphi(y^{\Pi}) < 0$

  $\Pi' := \text{greedy-moving}(\mathsf{move}_{aU}[\Pi])$

else

  $\Pi' := \Pi$

Greedy moving using the technique of Kernighan and Lin:

- Both algorithms discussed above terminate as soon as no transformation (join and move, resp.) leads to a decomposition with strictly lower cost.
- This can be sub-optimal in case transformations that increase the cost at one point in the recursion can lead to transformations that decrease the cost at later points in the recursion and the decrease overcompensates the increase.
- A generalization of local search introduced by Kernighan and Lin (1970) can escape such sub-optimal fixed points.
- Its application to greedy moving (next slide) builds a sequence of moves and then carries out the first t moves whose cumulative decrease in cost is optimal.

$\Pi' = \text{greedy-moving-kl}(\Pi)$

$\Pi_0 := \Pi$
$\delta_0 := 0$
$A_0 := A$
$t := 0$
repeat (build sequence of moves)
    choose $(a_t, U_t) \in \underset{a \in A_t, U \in \mathcal{N}_a}{\operatorname{argmin}} \varphi(y^{\mathsf{move}_{aU}[\Pi_t]}) - \varphi(y^{\Pi_t})$
    $\Pi_{t+1} := \mathsf{move}_{a_t U_t}[\Pi_t]$
    $\delta_{t+1} := \varphi(y^{\Pi_{t+1}}) - \varphi(y^{\Pi_t})$
    $A_{t+1} := A_t \setminus \{a_t\}$ (move a_t only once)
    $t := t + 1$
until $A_t = \emptyset$

$\hat{t} := \min \underset{t' \in \{0,\dots,|A|\}}{\operatorname{argmin}} \sum_{\tau=0}^{t'} \delta_\tau$ (choose sub-sequence)

if $\sum_{\tau=0}^{\hat{t}} \delta_\tau < 0$
    $\Pi' := \text{greedy-moving-kl}(\Pi_{\hat{t}})$ (recurse)
else
    $\Pi' := \Pi$ (terminate)

Summary:

- Image decomposition assumes the form of the minimum cost multicut problem with cost coefficients learned from data.
- Local search algorithms for this problem include greedy joining, greedy moving and greedy moving using the technique of Kernighan and Lin.